

On the Long-Repetition-Free 2-Colorability of Trees

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Abstract

A word $\bar{w} = \bar{u}\bar{u}$ is a *long square* if \bar{u} is of length at least 3; a word \bar{w} is *long-square-free* if \bar{w} contains no sub-word that is a long square. We can use words to generate graph colorings; a graph coloring is called *long-repetition-free* if the word formed by the coloring of each path in the graph is long-square-free. Every rooted tree of radius less than or equal to seven is long-repetition-free two-colorable. There exists a class of trees which are not long-repetition-free two-colorable.

Keywords: graph coloring; long-square-free; long-repetition-free

Introduction

The aim of this paper is to introduce the concepts of “long square” and “long repetition,” and to prove two main results regarding the long-repetition-free 2-colorability of trees. Most terminology relevant to this paper is provided, but the reader is encouraged to reference M. Lothaire [3] and Bondy and Murty [1] for more information regarding combinatorics on words and graph theory, respectively.

The investigation of words that avoid specific letter patterns (such as, for example, squares, cubes, or palindromes) has been a topic of interest in the mathematical sciences and the biological sciences since the early 20th century, beginning largely with the papers of Axel Thue (cf. [4], [5]). Thue proved that it is impossible to define an infinitely-long, square-free word over a binary alphabet; however, Thue proved that it *is* possible to define an infinitely-long, square-free word over a ternary alphabet, and Thue gave a definition for such a word (which today is called the Thue-Morse word). One can use such a word to color the vertices of a graph and, in so doing, create what we call repetition-free 3-colorings.

We say a word \bar{w} is a “long square” if $\bar{w} = \bar{u}\bar{u}$ for some word \bar{u} where $|\bar{u}| \geq 3$. Much is known regarding words that avoid squares generally (and graph colorings that avoid repetitions generally) - see Grytczuk [2] for a survey of such results; however, little is known about words that allow for squares of lengths 2 and 4.

In this article, we use two results regarding palindromes (in particular, “long palindromes,” or palindromes of length at least 4) to prove two main results regarding the long-repetition-free 2-colorability of trees. The first main

result asserts that every rooted tree of radius less than or equal to seven is long-repetition-free 2-colorable. The second main result answers the question of whether or not every tree is long-repetition-free 2-colorable. We construct a tree called a Tyler tree of height 8 and prove that it is not long-repetition-free 2-colorable.

1 Terminology and Notation

We begin by providing several useful definitions and detailing the notation that is used throughout this paper.

Definition 1.1. An *alphabet* is a finite set of elements called *letters*, and we typically denote alphabets by a single capital letter (for example, A). The set of all words over alphabet A is denoted A^* . We typically denote words of arbitrary length by a single lower-case letter with an overhead bar (for example, \bar{w}), and we typically denote letters (words of length 1) by a single lower-case letter with an overhead dot (for example, \dot{a}). We denote by ϵ the word of length 0, called the *empty word*. We denote the concatenation of words \bar{w} and \bar{v} by $\bar{w}\bar{v}$ or, if we wish to stress the significance of concatenation, by $\bar{w} \cdot \bar{v}$. If a word \bar{u} contains sub-word \bar{w} , we say $\bar{w} \leq \bar{u}$.

Definition 1.2. For word $\bar{w} = \dot{a}_1 \dots \dot{a}_n$ over alphabet A , we define the *reversal* of \bar{w} to be $\overleftarrow{\bar{w}} = \dot{a}_n \dots \dot{a}_1$. A *palindrome* is a word \bar{w} such that $\bar{w} = \overleftarrow{\bar{w}}$. We define a *long palindrome* to be a palindrome of length at least 4. Word \bar{w} is a *square* if there exists some non- ϵ word \bar{u} such that $\bar{w} = \bar{u}\bar{u}$. Word \bar{w} is *square-free* if \bar{w} contains no sub-word that is a square. Similarly, word $\bar{w} = \bar{u}\bar{u}$ is a *long square* if $|\bar{u}| \geq 3$. Word \bar{w} is *long-square-free* if \bar{w} contains no sub-word that is a long square.

Definition 1.3. Let $G = (V, E)$ be a simple, connected graph, and let $A = \{1, \dots, k\} \subset \mathbb{N}$. Graph G is said to be *long-repetition-free k -colorable* if there exists a coloring $\chi : V \rightarrow A$ such that, for every path $P = u_1 e_1 \dots e_{n-1} u_n \subseteq G$, the word $\chi(u_1) \dots \chi(u_n)$ is long-square-free.

Definition 1.4. Let $G = (V, E)$ be a connected graph. The *distance* between vertices $u, v \in V$ is defined to be the number of edges contained in the shortest path from u to v . The *eccentricity* of vertex v is the maximum distance between v and any other vertex in G . The *radius* of graph G is the minimum eccentricity of its vertices, and the *center* of graph G is the sub-graph induced by the vertices of minimum eccentricity.

Definition 1.5. A tree $T = (V, E)$ is called a *rooted tree* if one vertex has been distinguished to be called the *root* of T . The edges of a rooted tree are described with an implicit orientation away from the root vertex. In a rooted tree, the *parent* of a vertex v is the vertex adjacent to v on the path to the root vertex. A *child* (plural, *children*) of vertex v is a vertex of which v is the parent. A *descendant* of a vertex v is any vertex which is either a child of v or is, recursively, the descendant of any of the children of v . A *sibling* to a vertex v is any other vertex in V which has the same parent as v . The k^{th} generation of root r of tree T , denoted $G_k(T)$, is the set of descendant vertices of distance k from r . The *height* of T is the number of generations in T from r , with the convention that $G_0(T) = \{r\}$.

In particular, it is of note that if $T = (V, E)$ is a rooted tree, then for root $r \in T$ and vertex $v \in V$ there exists a unique paths from r to v and from v to r . As a matter of notation, it is perhaps noteworthy that while the k^{th} generation from r is of course dependent on our choice of r , we simply notate this $G_k(T)$ since, for this paper, the particular vertex distinguished as the root is not particularly relevant and furthermore does not change once the root has been chosen.

Definition 1.6. The *Tyler tree of height n* , denoted $\mathbb{T}_n = (\mathbb{V}_n, \mathbb{E}_n)$, is the rooted tree with root r in which each vertex in the j^{th} generation from r has $2^{n-j} + 1$ children, for every $0 \leq j \leq n - 1$. Computationally, $|\mathbb{V}_n| = 1 + \sum_{j=0}^{n-1} \prod_{k=n-j}^n (2^k + 1)$. A *binary tree* of height n has at most $2^{n+1} - 1$ vertices. The reader can verify that the Tyler tree of height n contains $2^n + 1$ Tyler trees of height $n - 1$.

2 A Class of Trees that is Long-Repetition-Free 2-Colorable

The goal of the present section is to prove the first of our two main results: Proposition 2.2, which asserts that every rooted tree of radius less than or equal to 7 can be long-repetition-free 2-colored.

In order to prove Proposition 2.2, we rely on Lemma 2.1 which we now state and prove.

Lemma 2.1. *Let $A = \{\dot{a}, \dot{b}\}$, and let $\bar{w} \in A^*$. If $\bar{w} = \dot{a}\bar{x}\dot{b}\dot{a}$ for some $\bar{x} \in A^*$, then \bar{w} contains a long palindrome.*

Proof. We consider two cases: case (1), if \bar{x} contains no square of length 2, and case (2), if \bar{x} contains at least one square of length 2.

Suppose first that \bar{x} contains no square of length 2. If $|\bar{x}| \leq 1$, then clearly we have the desired long palindrome. If $|\bar{x}| > 2$, then since by assumption \bar{x} contains no square of length 2, it must be the case that \bar{x} is an alternating sequence of \dot{a} 's and \dot{b} 's. It follows that \bar{x} must contain a long palindrome of the form $\dot{a}\dot{b}\dot{b}\dot{a}$ or $\dot{a}\dot{b}\dot{a}\dot{b}$.

Suppose next that $\dot{y}\dot{y} \leq \bar{x}$ for some $\dot{y} \in A$. It follows that, for some $\dot{l}_1, \dot{l}_2, \dot{r}_1, \dot{r}_2 \in A$, we have $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 \leq \bar{w}$. If $\dot{l}_2 = \dot{r}_1$, then we have the desired long palindrome. Suppose $\dot{l}_2 \neq \dot{r}_1$; then either $\dot{l}_2 = \dot{y} \neq \dot{r}_1$ or $\dot{l}_2 \neq \dot{y} = \dot{r}_1$.

Suppose first that $\dot{l}_2 = \dot{y} \neq \dot{r}_1$, from which follows $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{l}_1\dot{l}_1\dot{y}\dot{y}\dot{r}_1\dot{r}_2 \leq \bar{w}$. If $\dot{l}_1 = \dot{r}_1$, then we have the desired long palindrome; otherwise, we have $\dot{y}\dot{y}\dot{y}\dot{y}\dot{r}_1\dot{r}_2 \leq \bar{w}$ since $\dot{l}_2 = \dot{y} \neq \dot{r}_1$ by assumption, which contains the desired long palindrome.

Now suppose $\dot{l}_2 \neq \dot{y} = \dot{r}_1$, from which follows $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_2 \leq \bar{w}$. Either $\dot{l}_1 = \dot{y}$ or $\dot{l}_1 = \dot{l}_2$. In the former case, then either we have $\dot{r}_2 = \dot{y}$, in which case we have $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{y}\dot{l}_2\dot{y}\dot{y}\dot{y}$, or we have $\dot{r}_2 = \dot{l}_2$, in which case we have $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{y}\dot{l}_2\dot{y}\dot{y}\dot{l}_2$; in either case, we have the desired long palindrome. In the latter case where $\dot{l}_1 = \dot{l}_2$, then either we have $\dot{r}_2 = \dot{l}_2$ in which case we have $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{l}_2\dot{l}_2\dot{y}\dot{y}\dot{l}_2$, or we have $\dot{r}_2 = \dot{y}$, in which case we have $\dot{l}_1\dot{l}_2\dot{y}\dot{y}\dot{r}_1\dot{r}_2 = \dot{l}_2\dot{l}_2\dot{y}\dot{y}\dot{y}$; in either case, we have the desired long palindrome. ■

With Lemma 2.1 in hand, we move to state and prove the first main result of this paper.

Proposition 2.2. *Every rooted tree of radius less than or equal to 7 is long-repetition-free 2-colorable.*

Proof. Let $T = (V, E)$ be a rooted tree of radius less than or equal to 7. Let $v \in V$ be a vertex in the center of T , and designate v as the root. Let j be the number of generations of v , and note that $j \leq 7$. Let $\bar{a} = \dot{a}_1 \dots \dot{a}_7 = 00010111$, and let $\phi : V \rightarrow \{0, 1\}$ be the coloring defined by $\phi(u) = \dot{a}_i$, for every vertex u in the i^{th} generation from v and for every $1 \leq i \leq j$. We prove that ϕ is a long-repetition-free 2-coloring.

By way of contradiction, suppose there exists a path $P = p_1 \dots p_{2k} \in T$ for $k \geq 3$ such that, for words $\bar{u}' = \phi(p_1) \dots \phi(p_k)$ and $\bar{u}'' = \phi(p_{k+1}) \dots \phi(p_{2k})$, we have $\bar{u}' = \bar{u}''$. Since P contains a long repetition and since $\bar{a} = 00010111$ is long-square-free, P cannot be a path of vertices each from different generations. So, there exists some vertex $p_j \in P$ such that p_{j-1} and p_{j+1} are in the same generation; however, it must be the case that $p_j = v$, for in no other way can adjacent vertices of p_j be in the same generation. It follows that p_{j-1} and p_{j+1} are in the first generation from v , and the word $\bar{u}'\bar{u}''$ is of the form $\bar{u}'\bar{u}'' = \dot{a}_{j-1} \dots \dot{a}_1 \dot{a}_0 \dot{a}_1 \dots \dot{a}_{j-1} \dot{a}_j \dots \dot{a}_{2k-j}$. Assume that $j \leq k$ (for the argument that follows is similar for $j > k$), and we consider the cases where $j < k$ and $j = k$.

Case 1: suppose $j < k$. Then $\dot{a}_1 \dot{a}_0 \dot{a}_1 \leq \bar{u}'$. Since $\bar{u}' = \bar{u}''$, we have $\dot{a}_1 \dot{a}_0 \dot{a}_1 \leq \bar{u}''$ as well. Thus, since $j < k$, we have a word of the form $\dot{a}_0 \dot{a}_1 \bar{x} \dot{a}_1 \dot{a}_0 \leq \bar{a}$, where $\bar{x} \in \{0, 1\}^*$. By Lemma 2.1, we find that \bar{a} contains a palindrome of length 4 or 5. However, $\bar{a} = 00010111$ has no palindrome of length greater than 3, which is a contradiction.

Case 2: suppose $j = k$. Then either $\bar{a} = \bar{u}'$ and $\bar{a} = \overleftarrow{\bar{u}}''$, or $\bar{a} = \overleftarrow{\bar{u}}'$ and $\bar{a} = \bar{u}''$. Assuming, without loss of generality, that the latter is true, it follows that \bar{u}'' (and therefore \bar{a}) is of the form $\bar{u}' = \dot{a}_0 \dot{a}_1 \dots \dot{a}_1 \dot{a}_0$. Thus it follows by Lemma 2.1 that $\bar{a} = 00010111$ contains a palindrome of length 4 or of length 5, which is a contradiction. Hence, ϕ is a long-repetition-free 2-coloring. ■

3 A Class of Trees that is Not Long-Repetition-Free 2-Colorable

One may ask whether or not every tree is long-repetition-free 2-colorable. In this section, we show that a certain class of trees require a minimum of 3 colors for a long-repetition-free coloring, making 3 colors the lowest bound for long-repetition-free colorings of trees. In order to gain this result, we first prove Lemma 3.1 and Lemma 3.2.

Lemma 3.1. *Every binary word of length at least 9 contains a long palindrome.*

Proof. It is clear that we need only consider binary words of length 9, for any binary words of length greater than 9 contains a binary word of length 9. Let $\dot{a}_1 \dot{a}_2 \dots \dot{a}_9$ be a word over alphabet $A = \{0, 1\}$. Define \dot{a}_i^c to be the binary

complement of \dot{a}_i ; more precisely, $\dot{a}_i^c \equiv \dot{a}_i + 1 \pmod{1}$ for every $i \leq 9$. The reader can verify that for $\dot{a}_1 \dots \dot{a}_9$ to be long-palindrome-free, the following conditions must hold:

- (i) If $\dot{a}_i = \dot{a}_{i+3}$, then $\dot{a}_{i+1} = \dot{a}_{i+2}^c$, for $1 \leq i \leq 6$;
- (ii) If $\dot{a}_i = \dot{a}_{i+4}$, then $\dot{a}_{i+1} = \dot{a}_{i+3}^c$, for $1 \leq i \leq 5$;
- (iii) If $\dot{a}_i = \dot{a}_{i+1}$, then $\dot{a}_{i+1} = \dot{a}_{i+2}^c$, for $2 \leq i \leq 6$.

Suppose without loss of generality that $\dot{a}_0 = 0$. We consider four cases. First, suppose $\dot{a}_0 = \dot{a}_4 = \dot{a}_5 = 0$. By (i), it follows that $\dot{a}_2 = \dot{a}_3^c$. Then $\dot{a}_2 = 1$ and $\dot{a}_3 = 0$, for otherwise we have long palindrome 00100. However, we now have no choice for \dot{a}_6 , for $\dot{a}_6 = 0$ yields long palindrome 0000, and $\dot{a}_6 = 1$ yields long palindrome 10001.

Suppose next that $\dot{a}_1 = \dot{a}_4 = \dot{a}_5^c = 0$. By (i), it follows that $\dot{a}_2 = \dot{a}_3^c$. Then $\dot{a}_2 = 0$ and $\dot{a}_3 = 1$, for otherwise we have long palindrome 1001. Then $\dot{a}_6 = 1$, for otherwise we have long palindrome 01010, and so $\dot{a}_7 = 0$, for otherwise we have long palindrome 0110. However, we now have no choice for \dot{a}_8 , for $\dot{a}_8 = 0$ yields long palindrome 01110, and $\dot{a}_8 = 1$ yields long palindrome 1111.

Suppose that $\dot{a}_1 = \dot{a}_4^c = \dot{a}_5^c = 0$. By (iii), $\dot{a}_6 = 0$, and it follows that $\dot{a}_3 = 1$ for otherwise we have long palindrome 0110. However, we now have no choice for \dot{a}_2 , for $\dot{a}_2 = 0$ yields long palindrome 01110, and $\dot{a}_2 = 1$ yields long palindrome 1111.

Finally, suppose that $\dot{a}_1 = \dot{a}_4^c = \dot{a}_5 = 0$. By (ii), $\dot{a}_2 = 1$, and it follows that $\dot{a}_3 = 0$ for otherwise we have long palindrome 0110. Then $\dot{a}_6 = 1$, for otherwise we have long palindrome 00100; $\dot{a}_7 = 1$, for otherwise we have long palindrome 01010; and $\dot{a}_8 = 1$, for otherwise we have 0110. However, we now have no choice for \dot{a}_9 , for $\dot{a}_9 = 0$ yields long palindrome 01110, and $\dot{a}_9 = 1$ yields long palindrome 1111. ■

Lemma 3.2. *Let \mathbb{T}_n denote the Tyler tree of height n with root r , and apply an arbitrary 2-coloring ϕ to \mathbb{T}_n . Then \mathbb{T}_n contains a binary sub-tree of height n , denoted \mathbb{B}_n , with root r such that, for every $u, v \in G_i(\mathbb{B}_n)$, $\phi(u) = \phi(v)$, for all $0 \leq i \leq n - 1$.*

Proof. We prove the result by induction on height n .

Tyler tree \mathbb{T}_1 contains exactly four vertices: root r and 3 children vertices adjacent to r . By the pigeonhole principle, at least 2 of these 3 children

vertices must be colored with the same color. Thus, \mathbb{T}_1 contains a binary tree of height 1 whose root is r and where all the vertices on the same level have the same color. See Figure 3.3 for an example of \mathbb{T}_1 and \mathbb{T}_2 each with an embedded binary tree colored by generations.

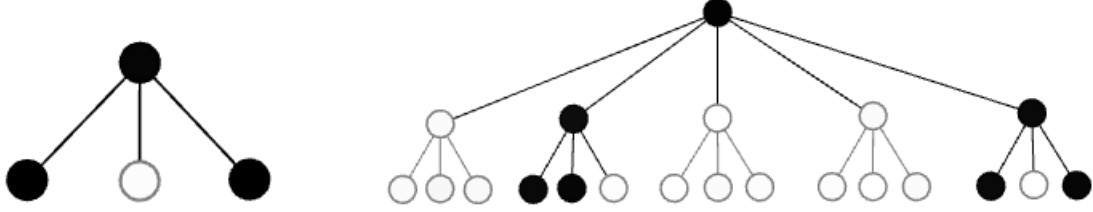


Figure 3.3. *The Tyler tree of height 1 (left) and the Tyler tree of height 2 (right) each contain a binary sub-tree in which every vertex in each generation from the root has the same color.*

Suppose every Tyler tree of height $k \leq n-1$ contains a binary sub-tree such that, for every $u, v \in G_i(\mathbb{B}_k)$, we have $\phi(u) = \phi(v)$, for every $0 \leq i \leq k-1$. Let \mathbb{T}_{k+1} denote the Tyler tree of height $k+1$ with root r , and apply arbitrary 2-coloring ϕ to \mathbb{T}_{k+1} . By definition, r has $2^{k+1} + 1$ children, and each of these children are the root of a Tyler tree of height k . Therefore, \mathbb{T}_{k+1} contains $2^{k+1} + 1$ Tyler trees of height k as sub-trees, and, by the inductive hypothesis, it follows that each of these Tyler trees of height k contain a binary sub-tree of height k such that, for every $u, v \in G_i(\mathbb{B}_k)$, we have $\phi(u) = \phi(v)$, for every $0 \leq i \leq k-1$.

Each of these $2^{k+1} + 1$ binary sub-trees has k generations, and, given the coloring constraints provided in the inductive hypothesis, there are exactly 2^k possible 2-colorings of these binary sub-trees. Attach to these binary sub-trees the root r . As there are 2 possible 2-colorings of r , there are a total of 2^{k+1} possible 2-colorings of this binary sub-tree of height $k+1$. Since there are $2^{k+1} + 1$ binary sub-trees of height k and only 2^{k+1} possible 2-colorings, it follows that at least two binary sub-trees of height k must be colored identically. The sub-tree induced by these two identically-colored binary trees of height k , when attached to r , form a binary tree of height $k+1$ in which, for every $u, v \in G_i(\mathbb{B}_{k+1})$, we have $\phi(u) = \phi(v)$, for all $0 \leq i \leq k$, thereby completing the proof. \blacksquare

Proposition 3.4. *If $n \geq 8$, then \mathbb{T}_n is not long-repetition-free 2-colorable.*

Proof. Let \mathbb{T}_8 denote the Tyler tree of height 8 with root r . We need only consider \mathbb{T}_8 to prove the result. Apply an arbitrary 2-coloring ϕ to \mathbb{T}_8 . By Lemma 3.2, \mathbb{T}_8 contains a binary sub-tree \mathbb{B}_8 in which, for each $u, v \in G_i(\mathbb{B}_8)$, we have $\phi(u) = \phi(v)$, for all $0 \leq i \leq 7$.

Pick vertices u_0, \dots, u_7 such that $u_i \in G_i(\mathbb{B}_8)$ for each $0 \leq i \leq 7$, and such that u_0, \dots, u_7 is a simple path. Let \dot{b}_i denote the letter associated with $\phi(u_i)$, and let $\bar{b} = \dot{b}_0 \dots \dot{b}_7$. By Lemma 3.1, \bar{b} contains a palindrome \bar{p} of length 4 or of length 5. Suppose $\bar{p} = \dot{b}_j \dots \dot{b}_{j+m}$, where $0 \leq j \leq 4$ and where $m = 3$ or $m = 4$. Consider the path $P = u_{j+m} \dots u_{j+1} u_j u'_{j+1} \dots u'_{j+m}$, where $u'_i \in G_i(\mathbb{B}_8)$ for each $0 \leq i \leq j + m$.

Let \bar{c} be the word associated with $\phi(P) = \phi(u_{j+m} \dots u_{j+1} u_j u'_{j+1} \dots u'_{j+m})$. If $|\bar{p}| = 4$, then either $\bar{c} = \dot{c}_1 \dot{c}_1 \dot{c}_1 \dot{c}_1$, or $\bar{c} = \dot{c}_1 \dot{c}_2 \dot{c}_2 \dot{c}_1$, where $c_i \in \{0, 1\}$. In either case, \bar{c} contains a long square. If $|\bar{p}| = 5$, then there are four possibilities: (1) $\bar{c} = \dot{c}_1 \dot{c}_1 \dot{c}_1 \dot{c}_1 \dot{c}_1$, (2) $\bar{c} = \dot{c}_1 \dot{c}_2 \dot{c}_1 \dot{c}_2 \dot{c}_1$, (3) $\bar{c} = \dot{c}_1 \dot{c}_1 \dot{c}_2 \dot{c}_1 \dot{c}_1$ or (4) $\bar{c} = \dot{c}_1 \dot{c}_2 \dot{c}_2 \dot{c}_1 \dot{c}_2$. In any of these four cases, \bar{c} contains a long square. Since \bar{c} contains a long square regardless of the length of \bar{p} , it follows that P contains a long repetition, and thus \mathbb{T}_8 is not long-repetition-free 2-colorable. \blacksquare

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References

- [1] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer, 2008.
- [2] Grytczuk. Nonrepetitive colorings of graphs - a survey. 2007:10, 2006.
- [3] M. Lothaire. *Combinatorics on Words*. Cambridge University Press, 1997.
- [4] A. Thue. Uber unendliche zeichenreihen. 7:1–22, 1906.

- [5] A. Thue. Über die gegenseitige lage gleicher teile gewisser zeichenreihen.
10:1–67, 1912.